Tfy-99.275 lecture 5

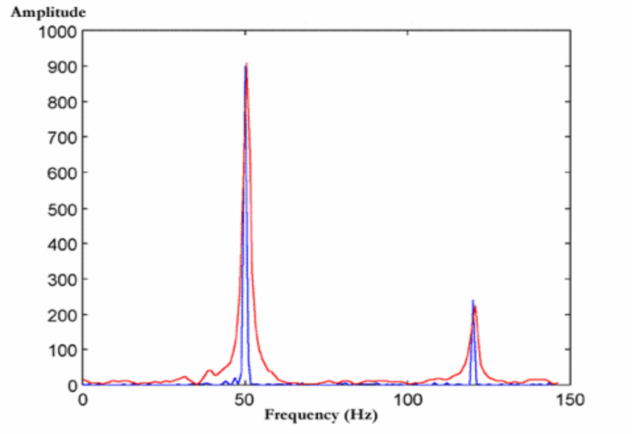
Time-frequency and
Wavelet Analysis

Time-frequency Analysis

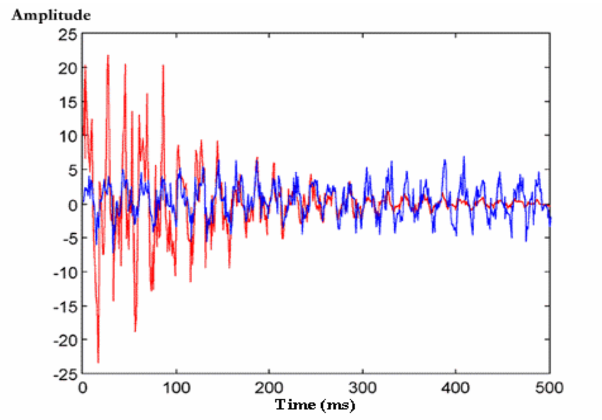
recall:

- in many physiological signals, especially frequency analysis is useful for interpreting and classifying the data
- the short-term Fourier transform (STFT) is a popular means to obtain a frequency representation of a segment of data
- however, the result of the STFT does not tell us *where* in that segment certain frequencies are present

two similar spectra...



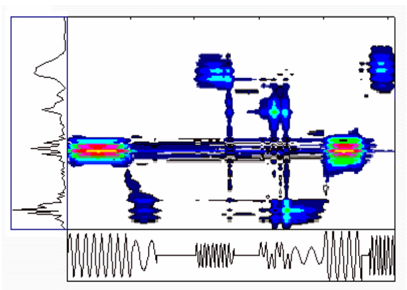
...can originate from totally different signals



Time-frequency methods

- if we make the windows, w, over which we calculate the STFT small enough we can follow the frequencies over time by letting the windows 'slide' along the time-axis

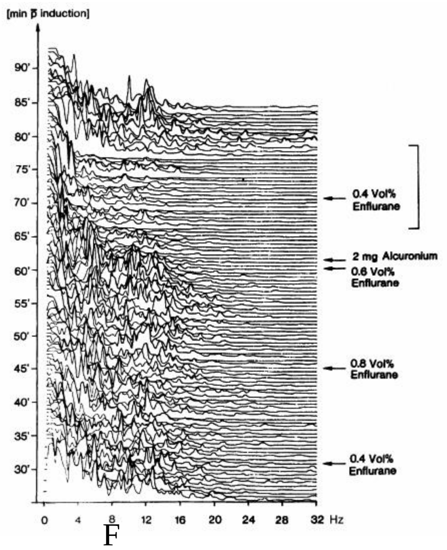
Time-frequency representations



F

t

t



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Problems with the STFT (1)

- shorter segments (narrower time-windows) lead to a better resolution in the time domain but also to an increase of the window width in the frequency domain and thus lead to poorer frequency resolution.
- the other way around: a good frequency resolution implies a large window width / poor resolution in the time domain (and thus requires strong assumptions about stationarity)

Problems with the STFT (2)

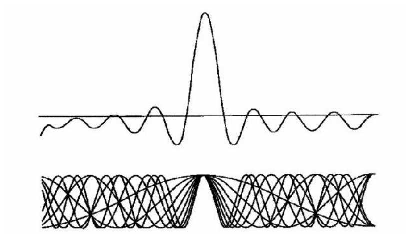
- the basis function for the STFT is the (complex) sinusoid; sinusoids are smooth and extending infinitely
- in biomedical signals we often encounter signals that contain 'sharp' features (e.g., epileptic spikes, ECG shape) that are typically localized in time
- sinusoids are not the best basis functions to represent such signals: small changes in the sine waves will produce changes everywhere in the time domain

Wavelets approach

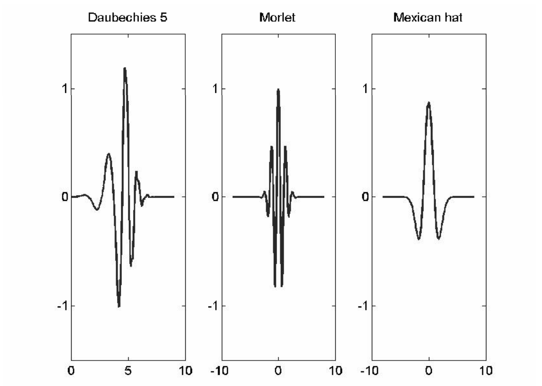
address these problems by

- making the time/frequency-window relation flexible according to our needs
- using basis functions that can both be localized in frequency (or: scale) and in time: *wavelets*

A sine wave as a basis function compared to a wavelet

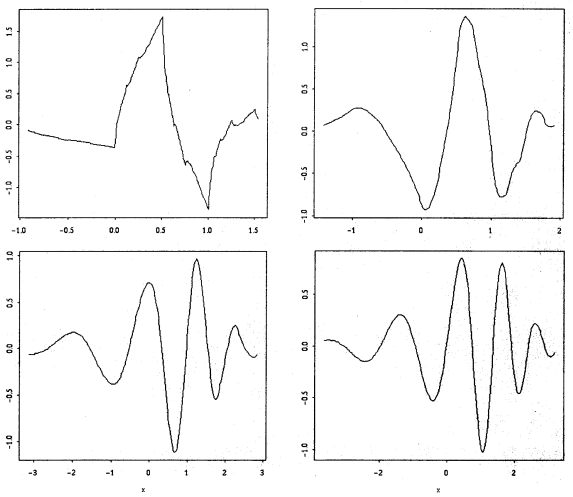


summation of sine waves (below) results in function in upper part (note: it extends to infinity)



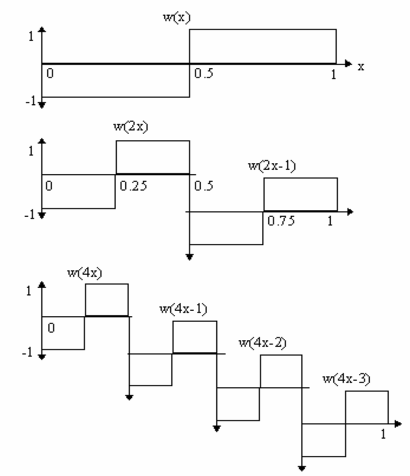
wavelets have different shapes, and do **not** extend to infinity

Some more examples of wavelets



note: the signals do **not** continue in a periodic fashion

Haar wavelets, levels 0,1,2



Basis functions using wavelets

define a 'mother' wavelet as *g*(*t*)

now we can make a basis to represent a signal *x*(*t*), by using compressed (or expanded) and shifted versions of *g*(*t*):

$$g\left(\frac{t-\tau}{a}\right)$$

the factor *a* controls the so-called dilations (or: scale factor/width) of the mother wavelet,
τ controls the translation/position of that wavelet

The Continuous Wavelet Transform (CWT) of *x*(*t*)

$$CWT_x(\tau, a) = \frac{1}{\sqrt{a}} \int x(t) \cdot g^*\left(\frac{t-\tau}{a}\right) \cdot dt$$

or, with change of variable, $t = at'$:

$$CWT_x(\tau, a) = \sqrt{a} \int x(at') \cdot g^*\left(t' - \frac{\tau}{a}\right) \cdot dt'$$

- a large scale factor, *a*, compresses the analysed function in time; it thus allows for examination of the signal over a wide interval
  (compare e.g., with the scale of a map; a large scale map may look at whole Europe, a small scale map may look at only (but detailedly) the university campus)

Scaling (1)

- for example if $g(t)$ were defined for $0 \le t \le 1$, then a scale factor, a, of 1 would lead to:

$$CWT_x(\tau,1) = \int_{\tau}^{\tau+1} x(t)\cdot g^*(t-\tau)\cdot dt$$

and a scale factor of 100 would yield:

$$CWT_x(\tau,100) = 10 \int_{\tau/100}^{1+\tau/100} x(100t)\cdot g^*\left(t-\frac{\tau}{100}\right)\cdot dt$$

- the signal under study is compressed, the width of the wavelet is not affected, the length of the integration interval stays the same, thus;large-scale features can be studied

Scaling (2)

- also, a large scale factor will yield a smaller window in the frequency domain, thus better frequency resolution.
- The opposite is also true; a small scale factor 'blows up' the signal and allows for representation of small-scale features, but it also leads to a poorer resolution in the frequency domain
- $g(t)$ can be regarded as a bandpass filtering function around some center frequency f_0

rewrite:

$$a = \frac{f}{f_0}$$

$$CWT_x\left(\tau, a = \frac{f}{f_0}\right) = \frac{1}{\sqrt{f/f_0}} \int x(t) \cdot g^*\left(\frac{t-\tau}{f/f_0}\right) \cdot dt$$

so, for larger center frequencies, we will have a smaller scale factor and larger bandwidth (poorer freq. resolution)

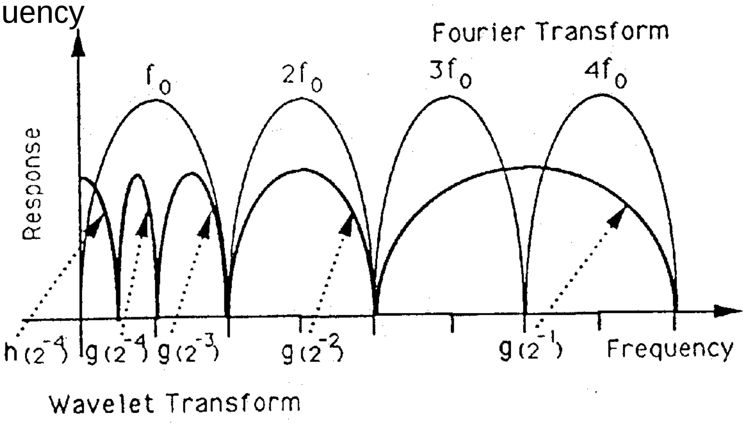
the short-term Fourier transform can be written as:

$$STFT(\tau, f) = \int_{-\infty}^{\infty} x(t) \cdot g^*(t-\tau) \cdot e^{-2\pi jft} dt$$

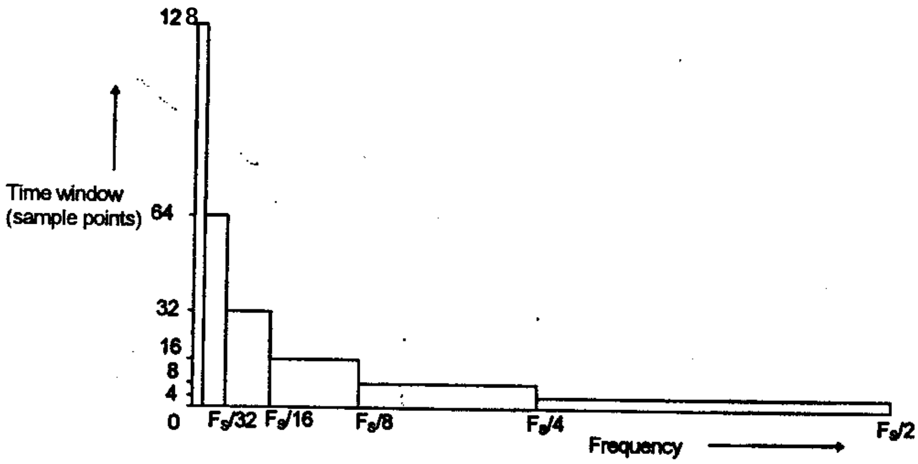
here, $g(t)$ is the sliding window

Frequency resolution:

- For the Fourier Transform the frequency resolution is constant across the entire spectrum, for the Wavelet Transform the resolution decreases (proportionally) with the frequency



Variable frequency and time resolution



Products of time and frequency uncertainties

$$\text{STFT}: \Delta t = \Delta t_g \quad ; \Delta f = \Delta f_0 \quad (\Delta t_g \text{ is time window})$$

$$\text{WT}: \quad \Delta t = a \cdot \Delta t_\psi ; \Delta f = \Delta f_\psi / a \quad (a \cdot \Delta t_\psi \text{ is time scale})$$

$$\Delta t_g \cdot \Delta f_0 = (a \cdot \Delta t_\psi)(\Delta f_\psi / a)$$

- The products are the same; with WT we have not gained anything with respect to ‘total uncertainty’ but we are able to tune the separate time and frequency uncertainties according to our interests

- for biomedical signals, that often have components that range from spikes (typically broadband, high frequency, signals) to slow wave components underlying the signal, this can be put to good use:
- we are typically interested in exactly when a spike occurs, we don't care so much what its exact frequency contents is: this can be well represented by small-scale CWTs that have good time resolution
- we usually don't need to locate the exact location of slow-wave components, a good estimation of the frequency content is of much more importance, use large scale CWTs

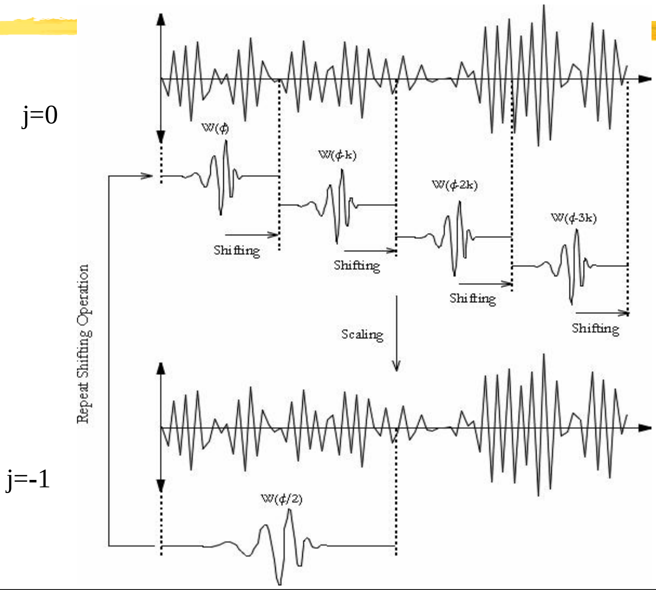
The Discrete Wavelet Transform (DWT)

- In the discrete case, 'scaling' implies changing the sampling rate: a larger scale is obtained by subsampling the signal
- Typically, a discrete wavelet function $\Psi(t)$ is scaled by values that are a power of 2

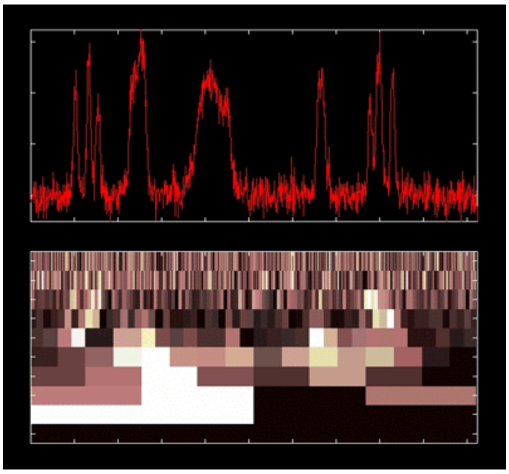
$$\psi_{2^j}(t) = 2^j \cdot \psi\left(2^j \cdot t\right)$$

- j is the scaling index, with j=0, -1, -2, ...

scaling and shifting using the DWT



10-level DWT of a transient signal



Multiresolution decomposition

- decomposition of a signal using a set of orthogonal wavelets
- orthogonality ensures a unique and complete representation of the signal,
- also the orthogonal complement provides some measure of the error in the representation

a given wavelet function $\phi(t)$ is dilated by a scale coeff. 2^j , translated by 2^{-j} and normalized by $\sqrt{2^{-j}}$

$$\sqrt{2^{-j}}\phi_{2^j}(t-2^{-j}n)$$

project the signal $f(t)$ onto the orthonormal basis $\mathbf{V}_{2^j}$ using an operator A_{2^j}

$$A_{2^j}f(t)=2^{-j}\sum_{n=-\infty}^{\infty}\left\langle f(u),\phi_{2^j}(u-2^{-j}n)\right\rangle\phi_{2^j}(t-2^{-j}n)$$

with

$$\left\langle f(u),\phi_{2^j}(u-2^{-j}n)\right\rangle=\int_{-\infty}^{\infty}f(u)\phi(u-2^{-j}n)du$$

2^j defines the resolution, A_{2^j} is the multiresolution operator that approximates the signal at resolution 2^j

Signals at lower resolutions can be obtained by repeated application of operator A_{2^j} $(-J \leq j \leq -1)$

- the part between <> is simply a convolution, and ϕ(t) the impulse response of the scaling function. The Fourier transforms of these of these functions are low-pass functions with successively smaller halfbands

- The convolution synthesizes the coarse signal at a resolution/scaling level j:

$$C_{2^j} f = \left\langle f(t), \phi_{2^j}(t - 2^{-j} n) \right\rangle$$

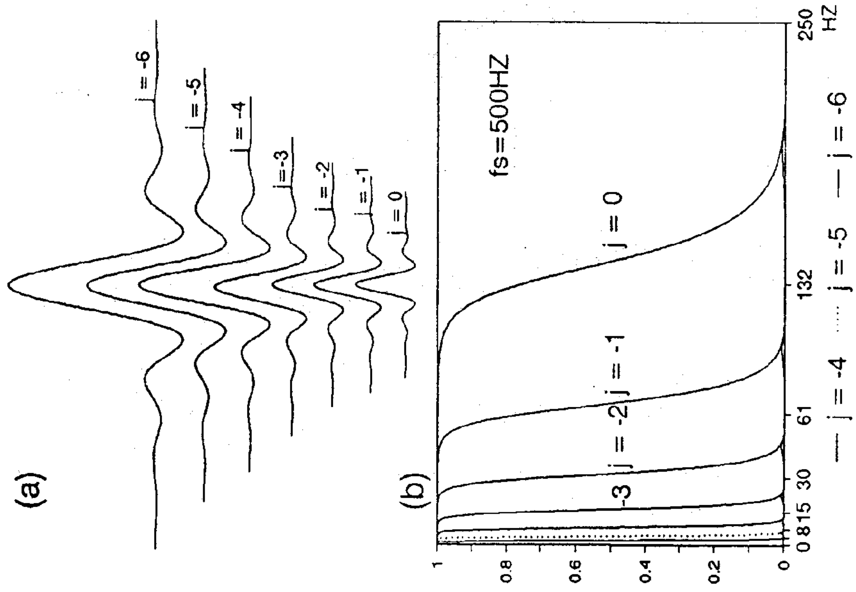


FIGURE 58.2 (*a*) Scaling functions $\phi(x)$ at various decreasing resolutions ($j = 0, -1, -2, \ldots, -6$). (*b*) Respective Fourier transforms of wavelet functions.

- now, try to express the basis function of one level of resolution, ϕ_{2^j} by a higher resolution $\phi_{2^{j+1}}$ Or, an orthogonal representation of the basis V_2 in terms of V_{2+1}

$$\phi_{2^j}(t-2^{-j}n) = 2^{-j-1}\sum_{k=-\infty}^{\infty}\left\langle \phi_{2^j}(u-2^{-j}n), \phi_{2^{j+1}}(u-2^{-j-1}k)\right\rangle \phi_{2^{j+1}}(t-2^{-j-1}k)$$

this conversion can be written as result of the application of a filter with impulse response *h(n)*

$$C_{2^j} f = \left\langle f(u), \phi_{2^j}(t-2^{-j}n)\right\rangle$$

$$= 2^{-j-1}\sum_{k=-\infty}^{\infty}\tilde{h}(2n-k)\left\langle f(u), \phi_{2^{j+1}}(t-2^{-j-1}k)\right\rangle$$

with $\tilde{h}(n) = h(-n)$

- If vector space V_2 can be regarded as the approximation of the the signal, then the orthogonal complement, O_2 , can be viewed upon as the error in that approximation
- This so-called 'error vector-space 'can in its turn be represented by orthogonal wavelet functions $\psi(x)$
- Similar to what we did for the scaling functions, we can use dilated wavelet functions to represent the orthogonal complement;

$$\sqrt{2^{-j}}\,\psi_{2^j}(t-2^{-j}n)$$

The Fourier transforms of these wavelets have band-pass characteristics; the relevant coefficients are obtained by the operator D:

$$D_{2^j} f = \left\langle f(t), \psi_{2^j}(t-2^{-j}n)\right\rangle$$

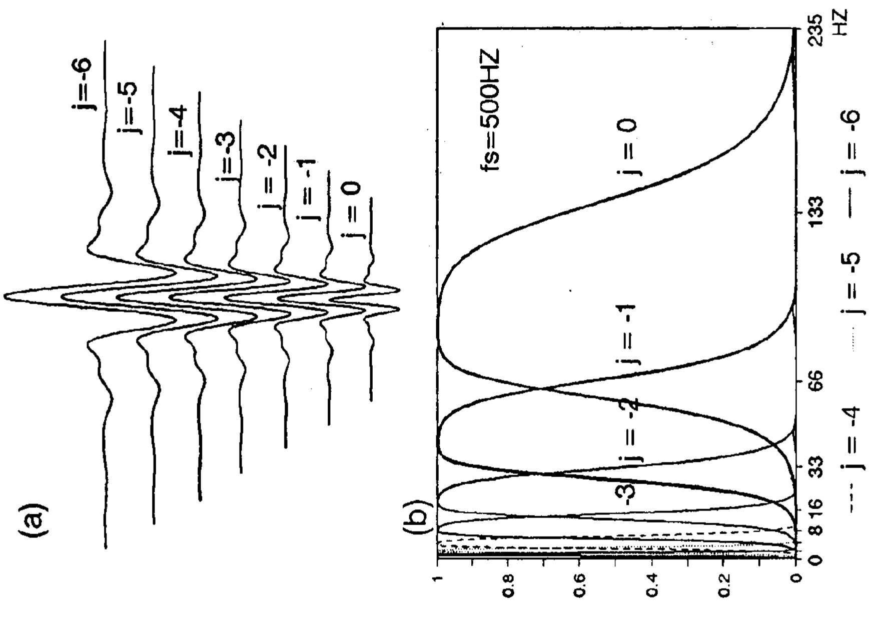
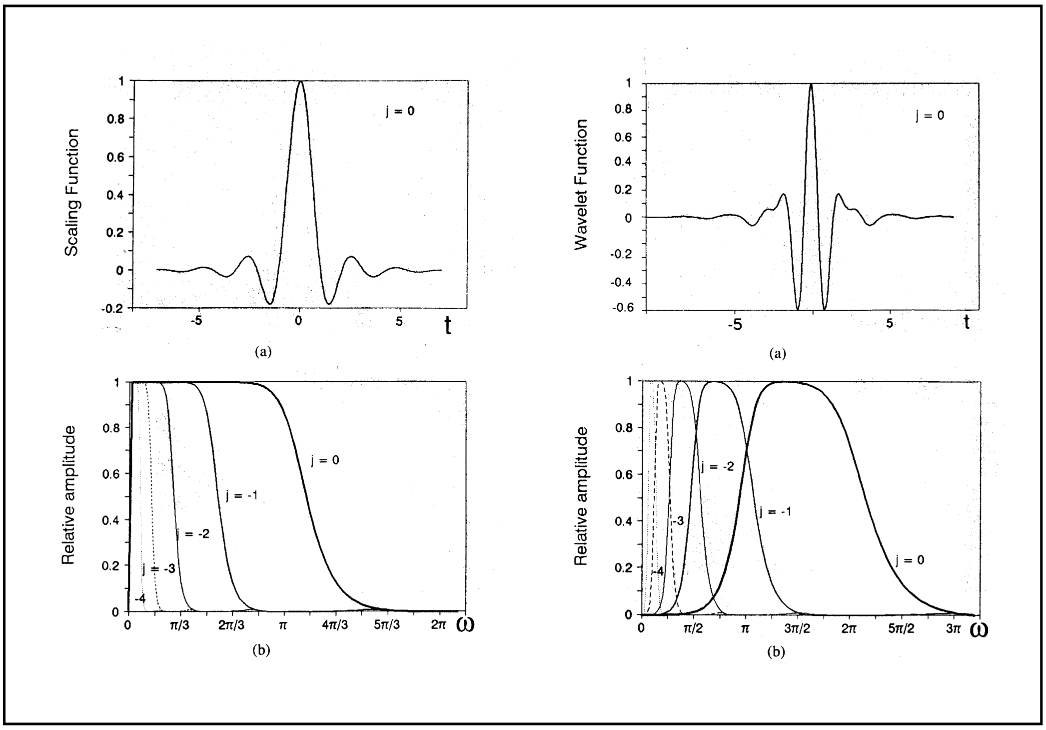
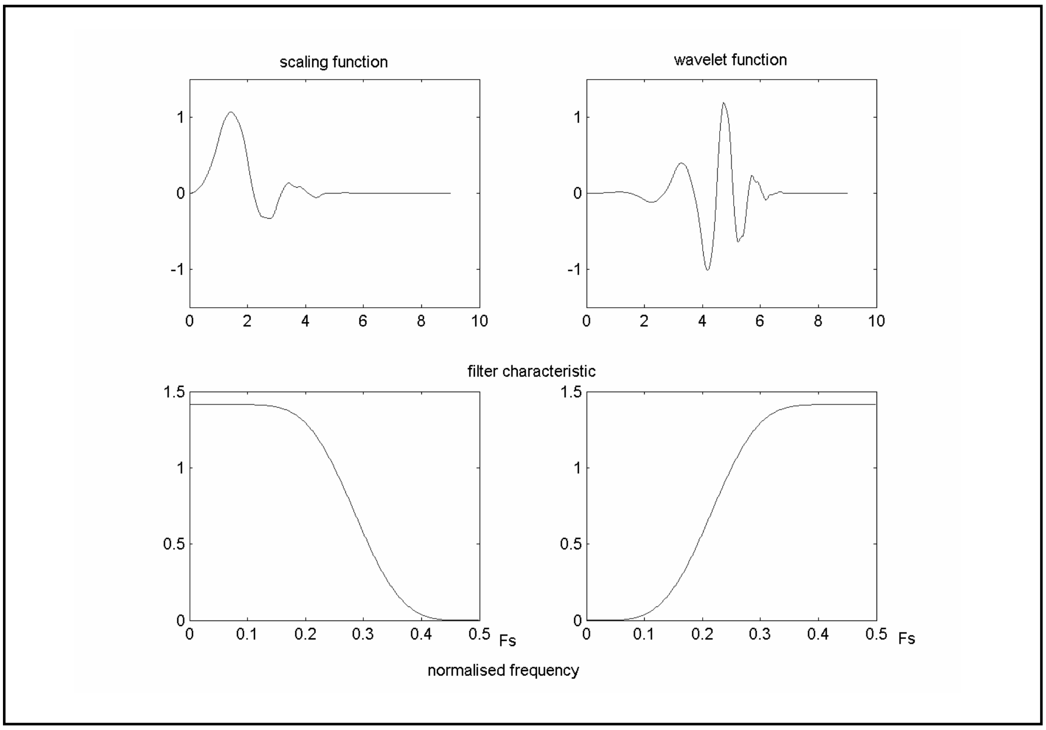


FIGURE 58.3 (*a*) Wavelet functions $\psi(x)$ at various decreasing resolutions ($j = 0, -1, -2, \ldots, -6$). (*b*) Respective Fourier transforms of wavelet functions.

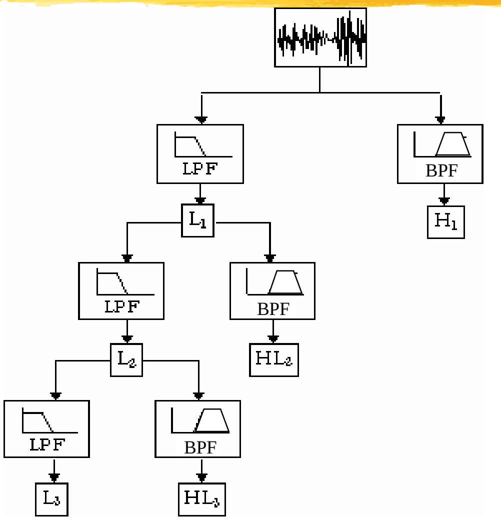
Algorithm for multiresolution wavelet decomposition

1. start with original signal $x(t)$ of N samples at resolution $j = 0$
2. convolve $x(t)$ with $\phi(t)$ to find $C_1 \cdot f$
3. find the coarse signal representation at successive resolution levels j=-1,-2, -3, … using the 'filter representation' formula for scaling functions and keep every other sample of the output
4. find detail signal representation at successive resolution levels j=-1,-2,-3,.. using the 'filter representation' formula for wavelet functions and keep every other sample of the output
5. decrease j and repeatedly calculate step 3 and 4 until j=-J

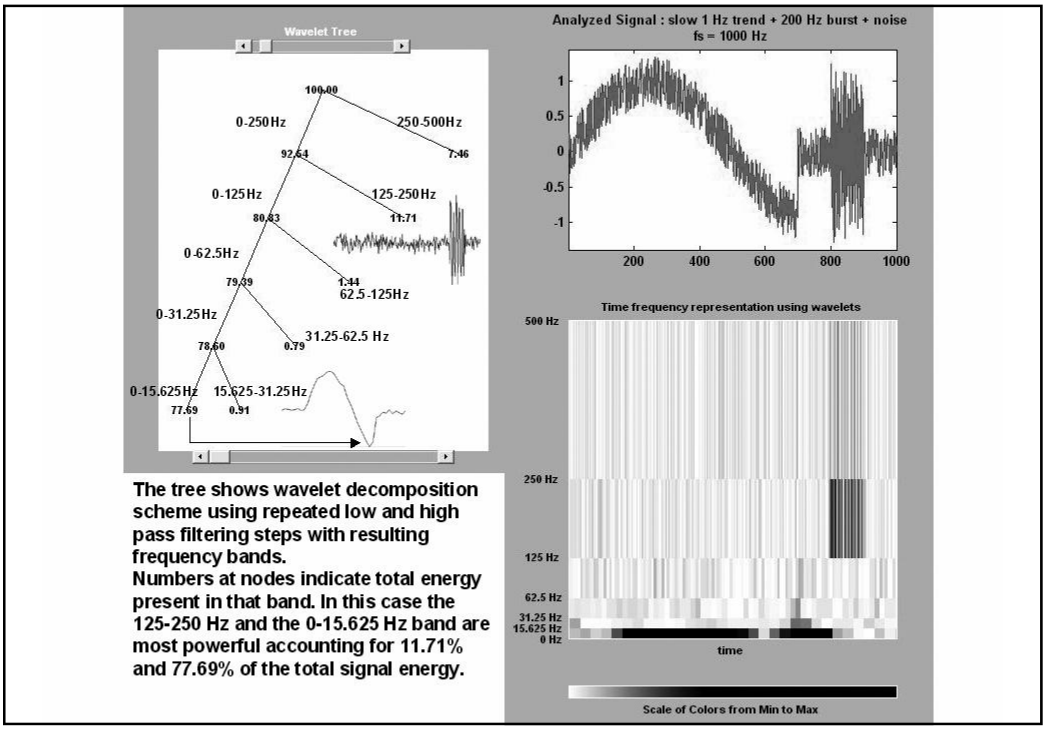

Scaling Function
j = 0
t
(a)
Wavelet Function
j = 0
t
(a)
Relative amplitude
j = 0
j = -1
j = -2
j = -3
-4
ω
(b)
Relative amplitude
j = -2
j = -1
-3
-4
j = 0
ω
(b)


scaling function
wavelet function
filter characteristic
Fs
Fs
normalised frequency

wavelet decomposition (filter banks)



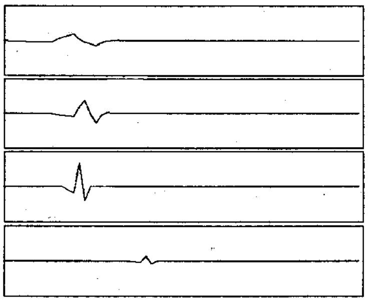
- Eventually, we have a set of scaling functions, ϕ, and a set of wavelet functions, ψ
- A signal can be represented by (decomposed in):
  - scaling functions that represent the coarse components, they are obtained by low-pass filtering the signal at successive steps
  - wavelet functions that represent detail components, they are obtained by band-pass filtering at successive steps
- each resolution level captures unique signal features



Data compression

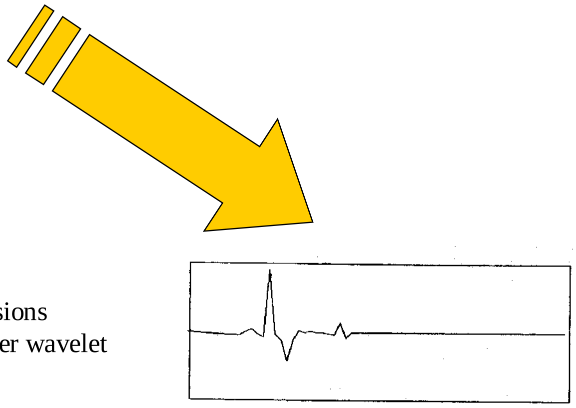
- Representation of a signal by using only the 'most important' coefficients (obtained e.g., by thresholding) can lead to considerable data compression; much less coefficients are needed to represent a signal than would be the case if Fourier coefficients were used, especially for signals with 'sharp' characteristics

Representation of an ECG cycle using only 4 wavelet coefficients



4 wavelet components of an ECG cycle.

note: the 4 curves above are all versions (dilated & shifted) of 1 single mother wavelet

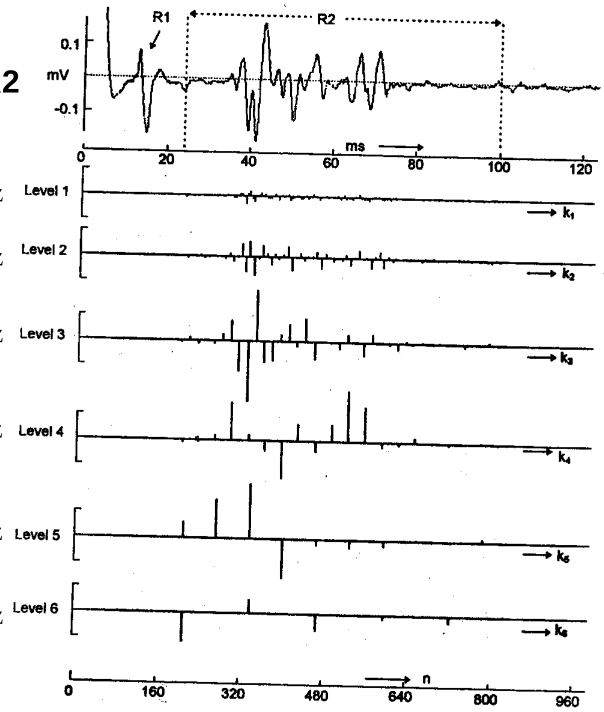


Sum of the four wavelets representing an ECG cycle.

Example difference between FT and WT

- A transient signal; recording of blink reflex by recording muscle activity after stimulation of supraorbital nerve (fs= 8000 Hz)
- Early reflex, R1, 5-15 ms after stimulus
- Late reflex, R2, 25-100 ms after stimulus
- Analysis of R2 is of considerable interest
- Wavelet decomposition using 6 Daubechies-4 type wavelets

wavelet transform of R2



f = 2000-4000 Hz

f = 1000-2000 Hz

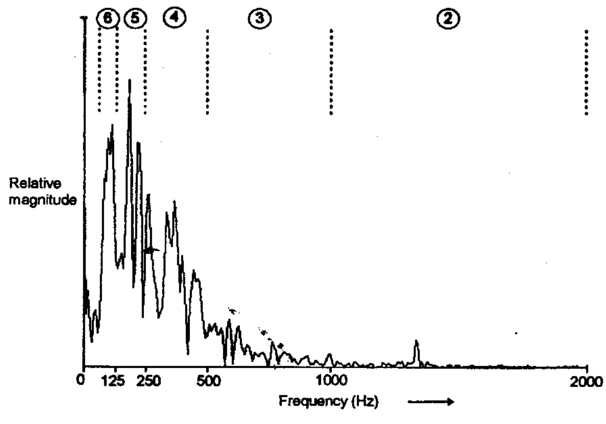
f = 500-1000 Hz

f = 250-500 Hz

f = 125-250 Hz

f = 62,5-125 Hz

Fourier transform of R2



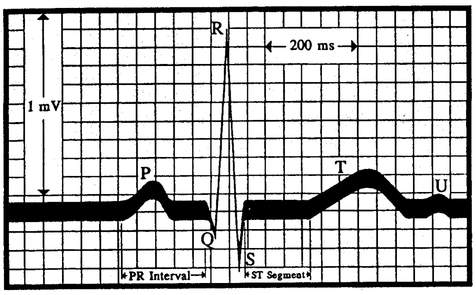
Fourier magnitude spectrum of R2 in the previous figure. Frequency bands corresponding to wavelet levels 2 to 6 are indicated.

Summary: Fourier vs Wavelets

- FT represents a signal as a combination of scaled and phase shifted sinusoids. Wavelet decomposition represents a signal as a combination of scaled wavelets; the specific shape of the wavelets used is application-dependent.
- DFT assumes the signal is periodic, and care must be taken to avoid misrepresentation in the frequency domain if the signal is in fact non-periodic. Wavelet decomposition is naturally able to represent non-periodic signals.
- FT acts on a block of data simultaneously; therefore information about the location of different components within this block is not available. In wavelet decomposition the time location of the wavelets is known from the position of the output coefficients.
- FT has equal time and frequency resolution for all components. Wavelet decomposition has a large time aperture, but closely spaced spectral resolution for slow components; and small time aperture but broad frequency resolution for fast components.

Applications

- Examination of ECG signals;
  low-frequency components are predominant in the P-complex and S-T segment;
  mid-to-high frequency components are present in the QRS complex



Detection of ischaemia

- normally done by looking at elevation/depression of the S-T segment. This requires analysis of frequencies down to as low as 0.05 Hz; this makes the method very sensitive to motion artefacts
- by monitoring the QRS complex we also can get relevant information while avoiding the influence of those artefacts. However, this requires us to have a good time-frequency representation

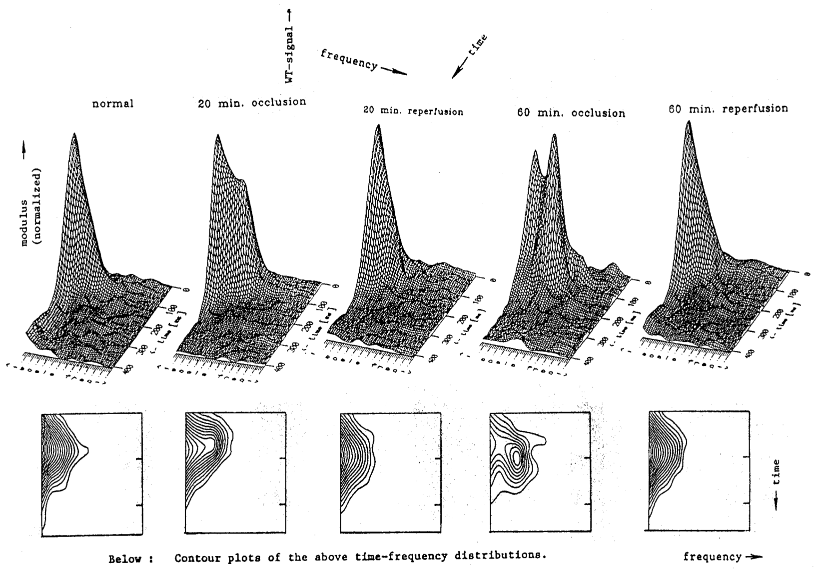


FIGURE 58.5 Time-frequency distributions of the vector magnitude of two ECG leads during five stages of a controlled animal experiment. The frequency scale is logarithmic, 16 to 200 Hz. The z axis represents the modulus (normalized) of the complex wavelet-transformed signal.

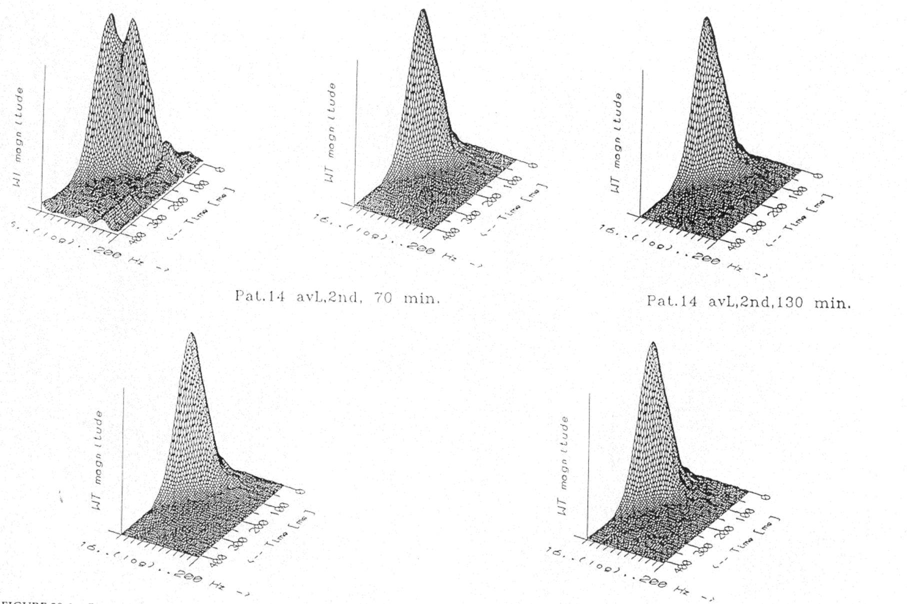


FIGURE 58.6 Contour plots of human ECG study using WT. Preangioplasty contour plot shows characteristic second hump. This disappears, as indicated by plots taken after two angioplasty treatments.

Analysis of late potentials

- Late potentials are high frequency components that appear in the S-T segment range. They may indicate dispersion of electrical activity of the cells in the heart and thus provide a substrate for production of heart arrhythmias.
- It is difficult to localize these high frequency components using FT representations.

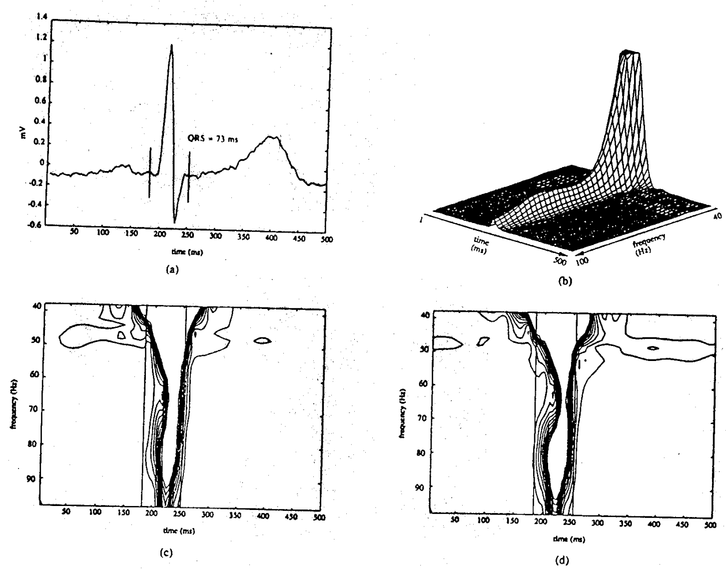


FIGURE 58.10 Patient with ventricular tachycardia diagnosis. (*a*) First beat. (*b*) 3-D representation of the modified WT for the first beat. (*c*) Contour plot of the modified WT for the first beat. (*d*) Contour plot of the modified WT for the second beat.

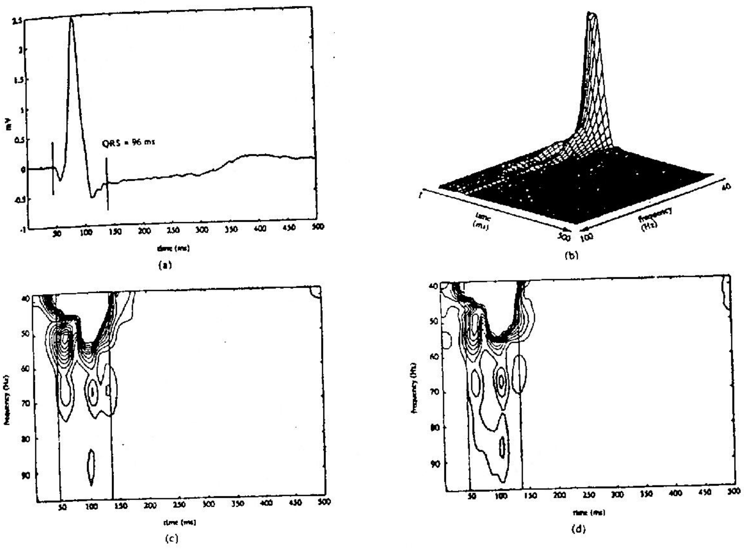


FIGURE 58.11 Healthy person. (*a*) First recorded beat. (*b*) 3-D representation of the modified WT for the first beat. (*c*) Contour plot of the modified WT for the first beat. (*d*) Contour plot of the modified WT for the second beat.

Neurologic Signal Processing

- Evoked Potential Monitoring
- Identification of features that are indicative of brain responses, e.g., monitoring of somatosensory evoked potentials (SEPs)
- Changing features in these signals can indicate brain injury (that might occur e.g., during risky operations) in early phases

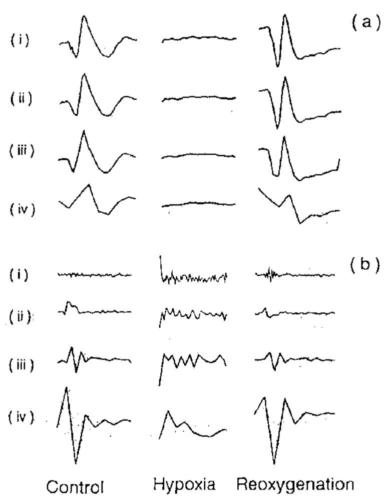


FIGURE 58.12 Coarse (a) and detail (b) components from somatosensory evoked potentials during normal, hypoxic, and reoxygenation phases of experiment.

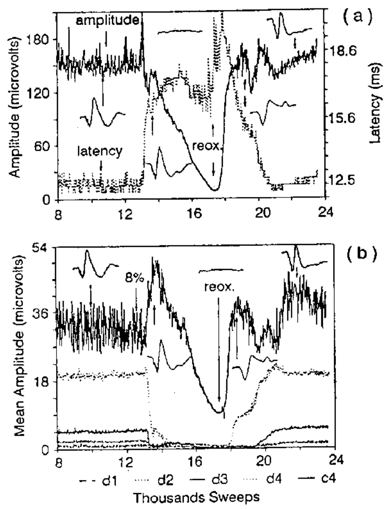


FIGURE 58.13 (a) Amplitude and latency of major evoked potential peak during control, hypoxic, and reoxygenation portions of experiment. (b) Mean amplitude of respective detail components during phases of experiment.

Localization of epileptic seizures

- In general it is difficult to exactly locate the onset of an epileptic seizure in the EEG signal.
- We would like to be able to detect 'sharp' features; 'detail component' are especially suited for this

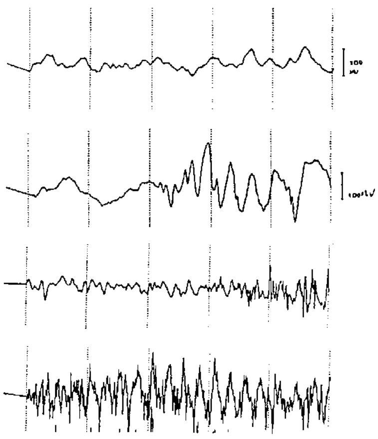


FIGURE 58.14 Example of epilepsy burst.

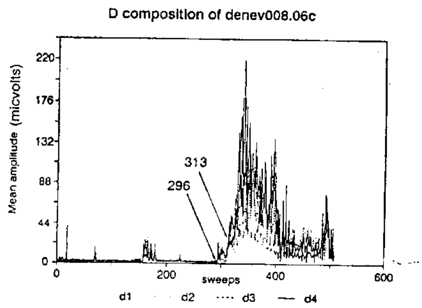
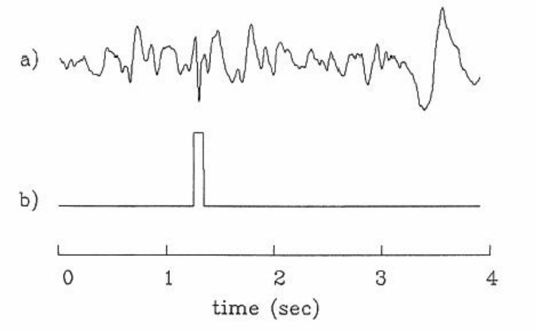
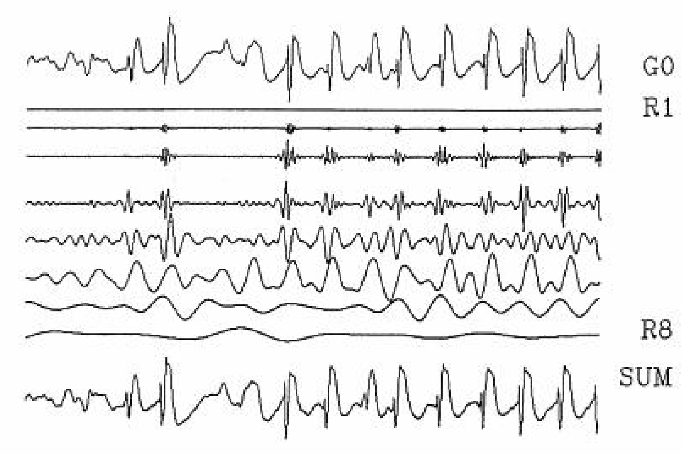


FIGURE 58.15 Localization of burst example with wavelet detail components.



Detection of an epileptiform spike by wavelet analysis: (a) EEG; (b) spike detection (much similar to a matched filtering approach)



Multiresolution decomposition wavelet analysis of a spike and wave episode. Traces R1 to R8 are the wavelet analysis of the signal in the top trace (G0). Wavelet dilation increases from R1 to R8. SUM shows reconstruction of the EEG by summing traces R1 to R8